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Change of base for measure spaces

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Abstract

In this paper, we use categorical disintegrations as an indexing notion. The program is to set up a framework for abstract indexing by measure spaces. We construct a pseudo-functorial pullback-like, though not universal, substitution and exhibit the Beck condition. Finally, we use this to understand the direct integral of Hilbert spaces in the context of indexed category theory.
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1. Introduction

The direct integral of Hilbert spaces $\int^{\oplus} H(x)\,\mathrm{d}\mu(x)$ has a measure-indexed aspect and a coproduct-like aspect. We would like to interpret this construction in the realm of indexed category theory to put it on a firm categorical footing. It is appropriate to set up a definition of abstract measurable family of Hilbert spaces so that $\int^{\oplus}$ becomes an indexed functor. Ideally, this would then be part of an indexed adjunction and would exhibit a universal property analogous to that for coproducts. It was noted in [6] that arriving at a left adjoint for an appropriate functorial notion of constant families Δ is too ambitious. This is not a serious flaw, however, because we can approximate the classical indexed category theory of [1] or [4] quite well. In [6], a framework was set up where a measurable Hilbert family was interpreted as a Hilbert space object in a certain topos. In this paper, we provide another framework where a measurable Hilbert family is interpreted in the context of slice categories.

Of fundamental importance in the indexing of sets by sets is the equivalence of categories $\underline{\mathbf{Set}}/I \cong \underline{\mathbf{Set}}^I$ for $I \in \underline{\mathbf{Set}}$. We explore a similar idea appropriately translated into a measure theoretic context as an approach to the problem of understanding indexing

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by measure spaces. The basic notion of *measurable family* will be a measure space over the measure space X.

In [5], we introduced two categories of measure spaces of finite measure relevant to indexing: **M0R** and **Disint**. The morphisms of **M0R** are measurable functions whose inverse image preserves null sets. These are called *measure zero reflecting* or *M0R*. The morphisms of **Disint**, called disintegrations, are measurable functions together with a family of measure structures on the fibres. There are some technical axioms imposed but the essence is to encapsulate the idea in Fubini's theorem: the measure of a set in the plane is obtained by integrating the measures of the fibres of that set. Disintegrations have a built-in self-indexed nature and we use this for our measure spaces over X. The premise is that an object of **Disint**$/X$ represents a good notion of X-family of measure spaces. For practical reasons (i.e. applications to the direct integral), **M0R** will be the base category for abstract indexing.

In this paper, we describe a pullback-like substitution to provide a good change of base for measure spaces, exhibit the Beck condition with respect to composition, and provide an application by discussing a framework for $\int^{\oplus}$.

It is well-known that for a topological space X, sheaves on X correspond to local homeomorphisms over X. The situation in measure theory is more complicated. The notions of *measurable sheaf* of [6] and *local homeomorphism* here seem to be quite different. It is not clear yet which is better for indexing purposes (in some sense, they are both equally good when X is a topological space). Indeed, this makes the situation in measure theory more interesting: there seem to be at least two non-trivial and non-equivalent indexing ideas.

2. Measure space background

Notation. Measurable spaces are denoted by ordered pairs, $(X,\mathscr{A})$, $(Y,\mathscr{B})$, etc., consisting of a set and a σ-algebra of subsets of that set. **Mble** denotes the category of measurable spaces and measurable functions. Measure spaces will be denoted by ordered triples, $(X,\mathscr{A},\mu)$, $(Y,\mathscr{B},\nu)$, etc., the first two items forming a measurable space and the third being a measure.

We will assume that singletons are measurable and that measure spaces have finite measure. These are usually called *finite measure spaces*. We do not assume completeness of measure. In particular, the product of two measure spaces (see the example below) is formed as the Cartesian product of the spaces with σ-algebra generated by the measurable rectangles and product measure. This measure is not completed.

Definition 2.1. A measurable function, $(X,\mathscr{A},\mu) \xrightarrow{f} (Y,\mathscr{B},\nu)$ is called *measure zero reflecting* or simply M0R if $\nu(B)=0 \Rightarrow \mu(f^{-1}(B))=0$. **M0R** is the category whose objects are finite measure spaces and whose morphisms are measure zero reflecting.

Definition 2.2. An object of **Disint** is a finite measure space. A morphism between two objects, $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$, is called a *disintegration*. It consists of an $(X, \mathscr{A}) \xrightarrow{f} (Y, \mathscr{B}) \in$ **Mble** and a family $(X_y, \mathscr{A}_y, \mu_y)_{y \in Y}$ of finite measure spaces, where $X_y := f^{-1}(y)$ and $\mathscr{A}_y = \{A \cap f^{-1}(y) \mid A \in \mathscr{A}\}$ subject to the axioms:
1. $\forall A \in \mathscr{A}$, the map $y \mapsto \mu_y(A \cap f^{-1}(y))$ is measurable and bounded and
2. $\forall A \in \mathscr{A}$, $\mu(A) = \int_Y \mu_y(A \cap f^{-1}(y))\, d\nu(y)$.

A disintegration is denoted by $(X, \mathscr{A}, \mu) \xrightarrow{f, \mu_y} (Y, \mathscr{B}, \nu)$. The identity on $(X, \mathscr{A}, \mu)$ is defined as $(X, \mathscr{A}, \mu) \xrightarrow{(1_X, \iota_x)} (X, \mathscr{A}, \mu)$ where 1_X is the identity function and ι_x is counting measure on the discrete σ-algebra on $\{x\}$ and for $(X, \mathscr{A}, \mu) \xrightarrow{(f, \mu_y)} (Y, \mathscr{B}, \nu) \xrightarrow{(g, \nu_z)} (Z, \mathscr{C}, \rho)$, the composite is defined as $(X, \mathscr{A}, \mu) \xrightarrow{(gf, \theta_z)} (Z, \mathscr{C}, \rho)$ where

$$\theta_z(E) := \int_{g^{-1}(z)} \mu_y(E \cap f^{-1}(y))\, d\nu_z(y) \quad \text{for } E \in \mathscr{E}_z = \{A \cap f^{-1}g^{-1}(z) \mid A \in \mathscr{A}\}.$$

For an extensive list of examples and basic properties, see [5]. Examples of disintegrations are also included in the substitution examples of Section 3.2. As alluded to in the Introduction, the paradigmatic example is:

Example. Let $(X, \mathscr{A}, \mu)$ and $(Y, \mathscr{B}, \nu)$ be two finite measure spaces and consider the projection onto the first factor,

$$(X \times Y, \mathscr{A} \otimes \mathscr{B}, \mu \otimes \nu) \xrightarrow{p} (X, \mathscr{A}, \mu),$$

where $\mathscr{A} \otimes \mathscr{B}$ is the σ-algebra generated by measurable rectangles and $p^{-1}(x) = \{x\} \times Y$.

Now, $(\mathscr{A} \otimes \mathscr{B})_x = \{D \cap p^{-1}(x) \mid D \in \mathscr{A} \otimes \mathscr{B}\} = \{\{x\} \times B \mid B \in \mathscr{B}\}$. Define $(\mu \times \nu)_x(\{x\} \times B) := \nu(B)$ and extend (but we may sometimes abuse notation and write $(\mu \otimes \nu)_x(D \cap p^{-1}(x)) := \nu(D_x)$ with D_x considered as an element of $\mathscr{B}$). Axiom 2 is a special case of Fubini's theorem.

Some useful results from [5, 6] are collected in the following:

Proposition 2.1. (i) **MOR** *and* **Disint** *have*

(a) *an initial object given by* $(\emptyset, \{\emptyset\}, 0)$,

(b) *a terminal object given by* $(1, 2, \textit{counting})$,

(c) *binary coproducts* $(X, \mathscr{A}, \mu) + (Y, \mathscr{B}, \nu) = (X + Y, \mathscr{A} + \mathscr{B}, \mu + \nu)$ (*the* σ*-algebra consists of sets of the form* $A + B$ *and* $(\mu + \nu)(A + B) = \mu(A) + \nu(B)$), *and*

(d) *these coproducts are disjoint.*

(ii) **MOR** *and* **Disint** *are monoidal categories. The unit is the terminal object and the* $\otimes$ *is the usual product of measure spaces.*

(iii) *There is a full functor* $\mathbf{Set}_f \xrightarrow{D}$ **Disint** *which puts a discrete measure space structure on a finite set.*

(iv) $(f, \mu_y) \in$ **Disint** $\Rightarrow f \in$ **MOR**.

Remark. (1) A M0R does not necessarily have a disintegration structure on it.
(2) **M0R** does not have products.

3. Substitution

3.1. Definition

In this section, we introduce a notion of substitution of a disintegration along a M0R. Consider a diagram:

$$\begin{array}{ccc} (Z,\mathscr{C},\rho) & \xrightarrow{\;r\;} & (Y,\mathscr{B},v) \\ \Big\downarrow{\scriptstyle (g,\rho x')} & & \Big\downarrow{\scriptstyle (f,v_x)} \\ (X',\mathscr{A}',\mu') & \xrightarrow[\phi]{} & (X,\mathscr{A},\mu) \end{array}$$

with $\phi \in$ **M0R**, $(f, v_x) \in$ **Disint**. We will describe Z, g, r, etc., and exhibit $(g, \rho_{x'})$ as a disintegration and r as a M0R to establish a pseudo-functorial change of base that satisfies the Beck condition with respect to composition. We will think of g as "the" substitution of the disintegration "f" along the M0R ϕ.

$(Z,\mathscr{C})$, g, and r are formed as the pullback of f along ϕ in **Mble** after an appropriate forgetting of measures. Thus, $Z=\sum_{x'\in X'} Y_{\phi(x')}$, where $Y_{\phi(x')} := f^{-1}(\phi(x'))$ (in general, T_k denotes the fibre over k when no confusion can arise). A typical element of Z is (y,x') where $x' \in X'$ and $y \in Y_{\phi(x')}$. The projections are $g(y,x')=x'$ and $r(y,x')=y$. Thus, $g^{-1}(x')=Y_{\phi(x')} \times \{x'\} \cong Y_{\phi(x')}$ and, for $A' \in \mathscr{A}'$,

$$g^{-1}(A') = \sum_{x'\in X'} K_{x'} \quad \text{where } K_{x'} = \begin{cases} Y_{\phi(x')}, & x' \in A', \\ \emptyset, & x' \notin A'. \end{cases}$$

On the other hand, $(r^{-1}(B))_{x'} = r^{-1}(B) \cap g^{-1}(x') = B \cap f^{-1}(\phi(x')) \times \{x'\}$. $\mathscr{C}$ is the σ-algebra generated by $g^{-1}(A')$, $r^{-1}(B)$ for $A' \in \mathscr{A}'$ and $B \in \mathscr{B}$.

Lemma 3.1. *Every $C \in \mathscr{C}$ is decomposable as*

$$\sum_{x'\in X'} C_{x'} := \sum_{x'\in X'} C \cap g^{-1}(x')$$

with $C_{x'} \in \mathscr{B}_{\phi(x')} \times \{x'\} = \{B \cap f^{-1}(\phi(x')) \times \{x'\} \mid B \in \mathscr{B}\}$.

Proof. Decomposable C's form a σ-algebra containing $g^{-1}(A')$ and $r^{-1}(B)$. □

Thus, $\mathscr{C}_{x'} = \{C \cap g^{-1}(x') \mid C \in \mathscr{C}\} \subseteq \mathscr{B}_{\phi(x')} \times \{x'\}$ for each $x' \in X'$. The other containment holds as well since for $B \cap f^{-1}(\phi(x')) \in \mathscr{B}_{\phi(x')}$, $(B \cap f^{-1}(\phi(x'))) \times \{x'\} =$

$r^{-1}(B)\cap g^{-1}(x')\in\mathscr{C}_{x'}$. And so, a typical element $C_{x'}\in\mathscr{C}_{x'}$ may be written as $B_{x'}\times\{x'\}$ with $B_{x'}\in\mathscr{B}_{\phi(x')}$. Define

$$\rho(C)=\rho\left(\sum_{x'\in X'} C_{x'}\right):=\int_{X'} v_{\phi(x')}(B_{x'})\,\mathrm{d}\mu'(x').$$

As with the product example above, we will abuse notation (identify $B\cap f^{-1}(\phi(x'))\times\{x'\}$ with $B\cap f^{-1}(\phi(x'))$ and $v_{\phi(x')}\otimes\imath_{\{x'\}}$, where $\imath_{x'}$ is the counting measure on $\{x'\}$, with $v_{\phi(x')}$) to write

$$\rho\left(\sum_{x'\in X'} C_{x'}n\right)=\int_{X'} v_{\phi(x')}(C_{x'})\,\mathrm{d}\mu'(x').$$

Lemma 3.2. *For $C\in\mathscr{C}, x\mapsto v_{\phi(x')}(C_{x'})$ is (measurable and) integrable.*

Proof. Let $C=g^{-1}(A')\cap r^{-1}(B)$. Then

$$v_{\phi(x')}((g^{-1}(A'\cap r^{-1}(B))_{x'})=v_{\phi(x')}(B\cap f^{-1}(\phi(x')))\cdot\chi_{A'}.$$

The second factor of the right-hand side is integrable since $\mu'(A')<\infty$. The first factor is integrable since it is the composite of $v_x(B\cap f^{-1}(x))$, which is positive and integrable, and $\phi(x')$, which is M0R (to show that the composite of a positive, integrable function with a M0R function is integrable, proceed through cases from step functions, through simple functions, to positive measurable functions). In particular, for $C=Z$, $v_{\phi(x')}(Z_{x'})$ is integrable. For any C, $C_{x'}\subseteq Z_{x'}$ so we need only show measurability but this is straightforward by exhibiting measurability remains valid under σ-algebra operations on the $g^{-1}(A')\cap r^{-1}(B)$'s. □

Lemma 3.3. *ρ is a finite measure on $\mathscr{C}$.*

Proof. For example, finiteness follows from Lemma 3.2. The rest is likewise straightforward. □

Proposition 3.1. *Given $f\in$ **Disint** and $\phi\in$ **M0R** then $(g,\rho_{x'})\in$ **Disint** where $\rho_{x'}(C_{x'}):=v_{\phi(x')}(C_{x'})$ (again, identify $\mathscr{C}_{x'}$ with $\mathscr{B}_{\phi(x')}$).*

Proof. We have shown that $\rho_{x'}$ is measurable and bounded. Axiom 2 follows by construction:

$$\rho\left(\sum_{x'\in X'} C_{x'}\right)=\int_{X'} v_{\phi(x')}(C_{x'})\,\mathrm{d}\mu'(x')=\int_{X'}\rho_{x'}(C_{x'})\,\mathrm{d}\mu'(x'). \qquad\square$$

Finally, we note that:

Proposition 3.2. $r\in$ **M0R**.

Proof. Let $B \in \mathscr{B}$ have $v(B) = 0$. We want to show $\rho(r^{-1}(B)) = 0$. Recall,

$$\begin{aligned}\rho(r^{-1}(B)) &= \int_{X'} v_{\phi(x')}(B \cap f^{-1}(\phi(x')))\,\mathrm{d}\mu'(x') \\ &=: \int_{X'} t_B(\phi(x'))\,\mathrm{d}\mu'(x'),\end{aligned}$$

where $t_B(x) = v_x(B \cap f^{-1}(x))$. Now, $0 = v(B) = \int_X v_x(B \cap f^{-1}(x))\,\mathrm{d}\mu(x)$, so we need only establish the following:

Lemma 3.4. *For $X' \xrightarrow{\phi} X \in$ <u>**M0R**</u> and $X \xrightarrow{t} \mathbf{R}^{\geq 0} \in$ <u>**Mble**</u>,*

$$\int_X t(x)\,\mathrm{d}\mu(x) = 0 \;\Rightarrow\; \int_{X'} (t \circ \phi)(x')\,\mathrm{d}\mu'(x') = 0.$$

Proof. Let t proceed through cases: characteristic function, simple function, then positive, measurable function. □

3.2. Examples

In this section, we provide a number of examples of substitution. The examples all follow the same basic format. Given a pair of morphisms

$$\begin{array}{ccc} & & (Y, \mathscr{B}, v) \\ & & \big\downarrow {\scriptstyle (f, v_x)} \\ (X', \mathscr{A}', \mu') & \xrightarrow[\phi]{} & (X, \mathscr{A}, \mu) \end{array}$$

with (f, v_x) a disintegration and ϕ a MOR, f and ϕ are varied to produce the examples (i.e., g and r are described).

Example 1. Product:

$$\begin{array}{ccc} (Z, \mathscr{C}, \rho) & \xrightarrow{r} & (Y, \mathscr{B}, v) \\ {\scriptstyle (g, \rho_{x'})} \big\downarrow & & \big\downarrow {\scriptstyle (!_Y, v)} \\ (X', \mathscr{A}', \mu') & \xrightarrow[!_{X'}]{} & (1, \mathscr{I}, \iota) \end{array}$$

$(1,\mathscr{I},\iota)$ denotes a one point measure space with element $\bigstar$, discrete σ-algebra, and counting measure. In this case,

$$Z=\sum_{x'\in X'} Y_\bigstar=\sum_{x'\in X'} Y=Y\times X'.$$

Furthermore,

$$g^{-1}(A')=\sum_{x'\in X'} K_{x'}=\sum_{x'\in A'} Y=Y\times A', r^{-1}(B)$$
$$=\{(y,x')\mid y\in B\cap !_Y{}^{-1}(!_{X'}(x'))\}=B\times X',$$

and $\mathscr{C}=\mathscr{B}\otimes\mathscr{A}'$. Let $C\in\mathscr{B}\otimes\mathscr{A}'$, then by Fubini's theorem, $C_{x'}=C\cap\{(y,t)\mid y\in Y, t=x'\}\in\mathscr{B}$ and

$$\rho\left(\sum_{x'\in X'} C_{x'}\right)=\int_{X'}\nu_{!_{X'}(x')}(C_{x'})\,\mathrm{d}\mu'(x')=\int_{X'}\nu(C_{x'})\,\mathrm{d}\mu'(x')(\nu\otimes\mu)(C)$$

(as usual, some identifications have been made). □

Remark. Z is a pullback object in Set and $(Z,\mathscr{C})$ is a pullback object in Mble but the above substitution square is not universal in M0R. The diagonal $(Y,\mathscr{B})\to(Y\times Y,\mathscr{B}\otimes\mathscr{B})$, which is not in M0R, manifests itself as a "universal arrow" of a special case of Example 1 with $(X',\mathscr{A}')=(Y,\mathscr{B})$ and $(X,\mathscr{A},\mu)=(1,\mathscr{I},\iota)$.

Example 2. Terminal object:

$$\begin{array}{ccc}
(Z,\mathscr{C},\rho) & \xrightarrow{\;!_Z\;} & (1,\mathscr{I},\iota)\\
\big\downarrow{\scriptstyle (g,\rho_{x'})} & & \big\downarrow{\scriptstyle (1,\iota)}\\
(X',\mathscr{A}',\mu') & \xrightarrow[\;!_{X'}\;]{} & (1,\mathscr{I},\iota)
\end{array}$$

Here, $g^{-1}(A')=\sum_{x'\in X'} K_{x'}\cong\sum_{x'\in A'} 1\cong A'$. $Z=\sum_{x'\in X'} 1\cong X'$ so that $\mathscr{C}\cong\mathscr{A}'$. Furthermore,

$$\rho\left(\sum_{x'\in A'} 1\right)=\int_{X'}\iota(K_{x'})\,\mathrm{d}\mu'(x')=\int_{A'}\iota(1)\,\mathrm{d}\mu'(x')=\iota(1)\cdot\mu'(A')=\mu'(A')$$

and so $\rho=\mu'$. In this example, $\mathscr{C}_{x'}\cong\{\emptyset,\{x'\}\}$ and $\rho_{x'}=$ the counting measure. Thus, $(g,\rho_{x'})$ is the identity (up to isomorphism).

In the rest of the examples, calculations are similar to those above and are omitted.

Example 3. Identity disintegration:

$$
\begin{array}{ccc}
(Z,\mathscr{C},\rho) & \xrightarrow{\ r\ } & (X,\mathscr{A},\mu) \\
{\scriptstyle (g,\rho x')}\Big\downarrow & & \Big\downarrow{\scriptstyle (1,\imath_x)} \\
(X',\mathscr{A}',\mu) & \xrightarrow[\phi]{} & (X,\mathscr{A},\mu)
\end{array}
$$

In this case, $Z \cong X'$, $\mathscr{C} \cong \mathscr{A}'$, and

$$\rho(A') = \rho\left(\sum_{x' \in A'} \phi(x')\right) = \int_{A'} \imath_{\phi(x')}(\phi(x'))\,\mathrm{d}\mu'(x') = \int_{A'} 1\,\mathrm{d}\mu'(x') = \mu'(A').$$

Example 4. Identity M0R:

$$
\begin{array}{ccc}
(Z,\mathscr{C},\rho) & \xrightarrow{\ r\ } & (Y,\mathscr{B},v) \\
{\scriptstyle (g,\rho x')}\Big\downarrow & & \Big\downarrow{\scriptstyle (f,v_x)} \\
(X,\mathscr{A},\mu) & \xrightarrow[1]{} & (X,\mathscr{A},\mu)
\end{array}
$$

In this case, $Z = Y$, $\mathscr{C} = \mathscr{B}$, and

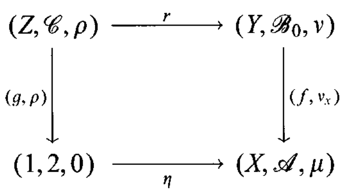
$$\rho(B) = \rho\left(\sum_{x \in X} B_x\right) = \int_X v_x(B_x)\,\mathrm{d}\mu(x) = \int_X v_x(B \cap f^{-1}(x))\,\mathrm{d}\mu(x) = v(B).$$

Example 5. Intersection: Let A_0 and A_1 be two measurable subsets of $(X,\mathscr{A},\mu)$.

$$
\begin{array}{ccc}
(Z,\mathscr{C},\rho) & \xrightarrow{\ r\ } & (A_0,\mathscr{A}_0,\mu_0) \\
{\scriptstyle (g,\rho x')}\Big\downarrow & & \Big\downarrow{\scriptstyle (i_0,\mu_{0_x})} \\
(A_1,\mathscr{A}_1,\mu_1) & \xrightarrow[i_1]{} & (X,\mathscr{A},\mu)
\end{array}
$$

Here, $Z \cong A_1 \cap A_0$, $\mathscr{C} \cong \mathscr{A}|_{A_1 \cap A_0} = \{A \cap A_1 \cap A_0 \mid A \in \mathscr{A}\}$, and $\rho(A \cap A_1) = \mu_1(A \cap A_1) = \mu(A \cap A_1)$ and $\rho(A \cap A_0) = \mu_0(A \cap A_0) = \mu(A \cap A_0)$.

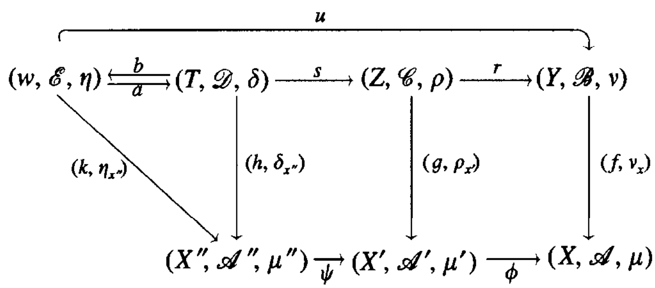
Example 6. Measure zero fibres:



Suppose $\eta(\bigstar)=x_0\in X$ which is not an atom. Then $Z=\sum_{\bigstar} Y\cong Y$ and $g^{-1}(\bigstar)=Z$ so $\mathscr{C}=\sigma(r^{-1}(B))\cong\mathscr{B}$. For each $B\in\mathscr{B}$, $\rho(B)=\int \nu_{x_0}(B\cap f^{-1}(x_0))\,\mathrm{d}0=0$. There is no "picking an element map" in **MOR** $((1,2,\text{counting})\to(X,\mathscr{A},\mu)$ is not MOR unless the element is an atom). Thus, fibres have measure zero (as they should).

3.3. *Pseudo-functoriality*

Let $(\phi)^*$ denote substitution along ϕ. Example 4 above shows that $1^*\cong 1$. In this section, we will show $(\phi\psi)^*\cong\psi^*\phi^*$. Consider the diagram



with $(g,\rho_{x'})$, $(h,\delta_{x''})$, and $(k,\eta_{x''})$ instances of substitution.

$W\cong T$ as sets (in **Set**, these are just pullbacks). We will have use of the explicit form of the isomorphism a and its inverse b: $W=\{(y,x'')\mid\phi\psi(x'')=f(y)\}$, $T=\{(z,x'')\mid\psi(x'')=x'=g(y,x')\}=\{((y,x'),x'')\mid\psi(x'')=x'$ and $\phi(x')=f(y)\}$, $W\xrightarrow{a}T$ is $(y,x'')\mapsto(y,\psi(x''),x'')$ and $T\xrightarrow{b}W$ is $(y,x',x'')\mapsto(y,x'')$.

The following proposition shows that a is a measurable equivalence (this means a and b are measurable and a is measure preserving: $\eta(a^{-1}(D))=\delta(D)$ which implies b is measure preserving; see [5]).

Proposition 3.3. *(1) $(W,\mathscr{E})\xrightarrow{a}(T,\mathscr{D})$ and $(T,\mathscr{D})\xrightarrow{b=a^{-1}}(W,\mathscr{E})$ are measurable.*
(2) $\eta(a^{-1}(D))=\delta(D)$, $\forall D\in\mathscr{D}$ and $\delta(b^{-1}(E))=\eta(E)$, for each $E\in\mathscr{E}$.

Proof. (1) For $D=h^{-1}(A'')\in\mathscr{D}$, $a^{-1}h^{-1}(A'')=k^{-1}(A'')\in\mathscr{E}$. The case $D=s^{-1}C$ breaks down into subcases: subcase $C=r^{-1}(B)$: $a^{-1}s^{-1}r^{-1}(B)=u^{-1}(B)\in\mathscr{E}$; subcase $C=g^{-1}(A')$: $a^{-1}s^{-1}g^{-1}(A')=a^{-1}h^{-1}\psi^{-1}(A')=k^{-1}\psi^{-1}(A')\in\mathscr{E}$ since $\psi^{-1}(A')\in\mathscr{A}''$. Next, note that inverse image preserves σ-algebra operations. The proof for b is similar.

(2) The basic case is $E=k^{-1}(A'')\cap u^{-1}(B)$:

$$\begin{aligned}\delta(b(k^{-1}(A'')\cap u^{-1}(B))) &= \delta(b^{-1}k^{-1}(A'')\cap b^{-1}u^{-1}(B))\\ &= \delta(h^{-1}(A'')\cap s^{-1}r^{-1}(B))\\ &= \int_{X''}\rho_{\psi(x'')}(r^{-1}(B)\cap g^{-1}(\psi(x'')))\cdot\chi_{A''}\,\mathrm{d}\mu''\\ &= \int_{X''}\mu_{\phi\psi(x'')}(B\cap f^{-1}(\phi\psi(x'')))\cdot\chi_{A''}\,\mathrm{d}\mu''\\ &= \eta(k^{-1}(A'')\cap u^{-1}(B)).\qquad\square\end{aligned}$$

Remark. We have actually shown a stronger result than needed for our purposes here. In fact, a is a measure-preserving isomorphism which implies an isomorphism in **M0R** which implies (see [5]) an isomorphism in **Disint**.

4. Substitution along a disintegration

4.1. *Characterization*

In this section, we will assume ϕ is also endowed with a disintegration structure and discuss substitution of a disintegration along a disintegration. Our goal is to prove that r is also equipped with a disintegration structure and a symmetry result: for $\phi\in$ **Disint**, $f^*(\phi)$ and $\phi^*(f)$ are measurably equivalent. We begin by giving a characterization of ρ that, fibrewise, it looks like the product measure. Consider

$$\begin{array}{ccc}(Z,\mathscr{C},\rho) & \xrightarrow{\;r\;} & (Y,\mathscr{B},\nu)\\ {\scriptstyle(g,\rho x')}\big\downarrow & & \big\downarrow{\scriptstyle(f,\nu_x)}\\ (X',\mathscr{A}',\mu') & \xrightarrow[(\phi,\mu'_x)]{} & (X,\mathscr{A},\mu)\end{array}$$

with $(\phi,\mu'_x)\in$ **Disint**. Let θ_x denote the composite of μ'_x and $\rho_{x'}$. Then

$$\theta_x(C\cap g^{-1}\phi^{-1}(x))=\int_{\phi^{-1}(x)}\rho_{x'}(C\cap g^{-1}(x'))\,\mathrm{d}\mu'_x(x').$$

We require a technical lemma (whose proof is straightforward). As usual, $Y_x:=f^{-1}(x); X'_x:=\phi^{-1}(x)$.

Lemma 4.1. (a) $g^{-1}\phi^{-1}(x)=Y_x\times X'_x$,
(b) $g^{-1}(A')\cap r^{-1}(B)\cap(Y_x\times X'_x)=(B\cap Y_x)\times(A'\cap X'_x)$,
(c) $g^{-1}(A'\cap\phi^{-1}(x))=Y_x\times(A'\cap X'_x)$.

Proposition 4.1. *For* $C\in\mathscr{C}$, $\rho(C)=\int_X(\nu_x\otimes\mu'_x)(C\cap Y_x\times X'_x)\,\mathrm{d}\mu(x)$.

Proof. Since θ_x is a disintegration,

$$\rho(C)=\int_X\theta_x(C\cap g^{-1}\phi^{-1}(x))\,\mathrm{d}\mu(x)=\int_X\int_{\phi^{-1}(x)}\rho_{x'}(C\cap g^{-1}(x'))\,\mathrm{d}\mu'_x(x')\,\mathrm{d}\mu(x).$$

For $C=g^{-1}(A')\cap r^{-1}(B)$, $\rho_{x'}(C\cap g^{-1}(x'))=\nu_{\phi(x')}(B\cap f^{-1}(\phi(x')))\cdot\chi_{A'}$, so

$$\begin{aligned}\rho(C)&=\int_X\int_{\phi^{-1}(x)}\nu_{\phi(x')}(B\cap f^{-1}(\phi(x')))\cdot\chi_{A'}\mathrm{d}\mu'_x(x')\,\mathrm{d}\mu(x)\\&=\int_X\nu_x(B\cap f^{-1}(x))\int_{\phi^{-1}(x)}\chi_{A'}\mathrm{d}\mu'_x(x')\,\mathrm{d}\mu(x)\\&=\int_X\nu_x(B\cap f^{-1}(x))\cdot\mu'_x(A'\cap\phi^{-1}(x))\,\mathrm{d}\mu(x)\\&=\int_X\nu_x(B\cap Y_x)\cdot\mu'_x(A'\cap X'_x)\,\mathrm{d}\mu(x)\\&=\int_X(\nu_x\otimes\mu'_x)((B\cap Y_x)\times(A'\cap X'_x))\,\mathrm{d}\mu(x)\\&=\int_X(\nu_x\otimes\mu'_x)(g^{-1}(A')\cap r^{-1}(B)\cap(Y_x\times X'_x))\,\mathrm{d}\mu(x).\qquad\square\end{aligned}$$

4.2. r is a disintegration

Next, we show that r is part of a disintegration and the following diagram commutes:

$$\begin{array}{ccc}(Z,\mathscr{C},\rho) & \xrightarrow{(r,\rho_y)} & (Y,\mathscr{B},\nu)\\ {\scriptstyle(g,\rho x')}\downarrow & & \downarrow{\scriptstyle(f,\nu_x)}\\ (X',\mathscr{A}',\mu') & \xrightarrow[(\phi,\mu'_x)]{} & (X,\mathscr{A},\mu)\end{array}$$

Write $(\phi g,\theta_x)$ and (fr,γ_x) for the composites. $\mathscr{C}_y:=\{y\}\times\mathscr{A}'_{f(y)}$ (for example, $r^{-1}(B)\cap r^{-1}(y)=\{y\}\times\phi^{-1}(f(y))$, if $y\in B$, and $g^{-1}(A')\cap r^{-1}(y)=\{y\}\times A'\cap\phi(f(y)))$. We define ρ_y using $\mu'_{f(y)}$ in analogy to $\rho_{x'}$. $\rho_y(C\cap r^{-1}(y)):=\mu'_{f(y)}(C\cap\phi^{-1}(f(y)))$ (or better: $\rho_y(g^{-1}(A')\cap r^{-1}(y)):=\mu'_{f(y)}(A'\cap\phi^{-1}(f(y)))$ and

$\rho_y(r^{-1}(B)\cap r^{-1}(y)) := \mu'_{f(y)}(\phi^{-1}(f(y))\cdot \chi_B)$. Again, that ρ_y is a bounded measurable function of y is exactly the same as for $\rho_{x'}$. It remains to show the second axiom $\rho(C)=\int_Y \rho_y(C\cap r^{-1}(y))\,\mathrm{d}\nu(y)$.

Lemma 4.2. *For (f,ν_x) a disintegration and $Y \xrightarrow{k} \mathbf{R}$ a positive, measurable function,*

$$\int_X \int_{f^{-1}(x)} k(y)\,\mathrm{d}\nu_x(y)\,\mathrm{d}\mu(x) = \int_Y k(y)\,\mathrm{d}\nu(y).$$

Proof. If $k=\chi_B$, the right-hand side is $\int_Y \chi_B\,\mathrm{d}\nu(y)=\nu(B)$ and the left-hand side is

$$\int_X \int_{f^{-1}(x)} \chi_B\,\mathrm{d}\nu_x(y)\,\mathrm{d}\mu(x) = \int_X \nu_x(B\cap f^{-1}(x))\,\mathrm{d}\mu(x) = \nu(B)$$

by axiom 2. Then proceed from simple functions to positive, measurable functions. □

Corollary. $\int_X \int_{f^{-1}(x)} \mu'_{f(y)}(C\cap \phi^{-1}(f(y)))\,\mathrm{d}\nu_x(y)\,\mathrm{d}\mu(x) = \int_Y \mu'_{f(y)}(C\cap\phi^{-1}(f(y)))\,\mathrm{d}\nu(y)$.

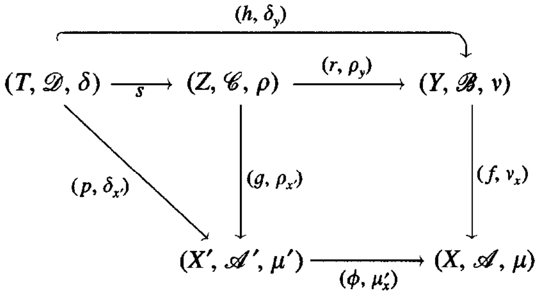
Proposition 4.2. $\int_Y \rho_y(C\cap r^{-1}(y))\,\mathrm{d}\nu(y)=\rho(C)$ *(axiom 2).*

Proof. Use the corollary and the proof of Proposition 4.1. For example, suppose $C=r^{-1}(B)$,

$$\begin{aligned}
\int_Y \rho_y(r^{-1}(B)\cap r^{-1}(y))\,\mathrm{d}\nu(y) &= \int_Y \mu'_{f(y)}(\phi^{-1}(f(y))\cdot\chi_B)\,\mathrm{d}\nu(y)\\
&= \int_X\int_{f^{-1}(x)} \mu'_{f(y)}(\phi^{-1}(f(y))\chi_B)\,\mathrm{d}\nu_x(y)\,\mathrm{d}\mu(x)\\
&= \int_X\int_{f^{-1}(x)} \mu'_x(\phi^{-1}(x)\chi_B)\,\mathrm{d}\nu_x(y)\mathrm{d}\mu(x)\\
&= \int_X\int_{f^{-1}(x)} \mu'_x(B\cap\phi^{-1}(x))\,\mathrm{d}\nu_x(y)\,\mathrm{d}\mu(x)\\
&= \int_X \mu'_x(B\cap\phi^{-1}(x))\left(\int_{f^{-1}(x)}\mathrm{d}\nu_x(y)\right)\mathrm{d}\mu(x)\\
&= \int_X \mu'_x(B\cap\phi^{-1}(x))\cdot\nu_x(f^{-1}(x))\,\mathrm{d}\mu(x)\\
&= \int_X (\nu_x\otimes\mu'_x)(Y_x\times(B\cap X'_x))\,\mathrm{d}\mu(x)\\
&= \rho(r^{-1}(B)). \qquad \square
\end{aligned}$$

4.3. *Symmetry*

Proposition 4.3. *In the diagram*



where $Z = \phi^*(f)$ *and* $T = f^*(\phi)$, $s(x', y) := (y, x')$ *is a measure equivalence.*

Proof. By the characterization above, $\rho(C) = \int_X (\nu_x \otimes \mu_x')\,(C \cap Y_x \times X_x')\,\mathrm{d}\mu(x)$ for all $C \in \mathscr{C}$ and $\delta(D) = \int_X (\mu_x' \otimes \nu_x)(D \cap X_x' \times Y_x)\,\mathrm{d}\mu(x)$ for all $D \in \mathscr{D}$. Thus, $\delta(s^{-1}(C)) = \rho(C)$ and $\rho(s(D)) = \delta(D)$. □

5. Composition

5.1. *Definition and basic properties*

For a disintegration $(X', \mathscr{A}', \mu') \xrightarrow{(\phi, \mu_x')} (X, \mathscr{A}, \mu)$,

$$\underline{\mathbf{Disint}}/X' \xrightarrow{\sum_\phi^D} \underline{\mathbf{Disint}}/X$$

denotes the precomposition with ϕ functor (precomposition in the case of $\underline{\mathbf{Mble}}/X$ will be denoted by $\sum_\phi^M$). In general, $\sum_\phi^D$ is not left adjoint to ϕ^*. Indeed, when ϕ is $! : X' \to 1$, $(\phi^* \sum_\phi^D)(X' \xrightarrow{1} X') = X' \times X' \to X'$; the unit at X' would be the diagonal which is not in $\underline{\mathbf{Disint}}$. However, since ϕ^* is the pullback in $\underline{\mathbf{Mble}}$,

$$\sum_\phi^M : \underline{\mathbf{Mble}}/X \rightleftarrows \underline{\mathbf{Mble}}/X' : \phi^*$$

is an adjunction: $\sum_\phi \dashv \phi^*$. For that matter, the category whose objects are disintegrations over a fixed measure space and whose morphisms are merely measurable functions making the appropriate triangle commute (i.e. the slice category but with merely measurable functions as morphisms), also has precomposition by ϕ left adjoint to substitution along ϕ.

$\sum_\phi^M$, being a left adjoint, preserves colimits. Two interesting properties of $\sum_\phi^D$ are given in the following two propositions:

Proposition 5.1. $\sum_\phi$*(initial) = initial.*

Proof. The initial object of $\underline{\textbf{Disint}}/X'$ is $(\emptyset,\{\emptyset,\}0) \xrightarrow{!_{X'}} (X',\mathscr{A}',\mu')$. Composing with $(\phi,\mathscr{A}'_x,\mu'_x)$ gives $(\emptyset,\{\emptyset\},0) \xrightarrow{!_X} (X,\mathscr{A},\mu)$, the initial object of $\underline{\textbf{Disint}}/X$. □

Proposition 5.2. $\sum_\phi$ *preserves binary coproducts.*

Proof. Let $(T,\mathscr{D},\delta) \xrightarrow{(h,\delta_{x'})} (X',\mathscr{A}',\mu')$ and $(S,\mathscr{C},\gamma) \xrightarrow{(g,\gamma_{x'})} (X',\mathscr{A}',\mu')$ be in $\underline{\textbf{Disint}}/X'$. The coproduct of S and T is $(S+T,\mathscr{C}+\mathscr{D},\gamma+\delta) \xrightarrow{(g+h,(\gamma+\delta)_x)} (X',\mathscr{A}',\mu')$ with $\mathscr{C}+\mathscr{D} := \{C+D \mid C\in\mathscr{C},\ D\in\mathscr{D}\}$, $(\gamma+\delta)(C+D) := \gamma(C)+\delta(D)$, and

$$(g+h)(t,i) := \begin{cases} g(t), & i=1,\\ h(t), & i=2.\end{cases}$$

Note that $(\mathscr{C}+\mathscr{D})_{x'} = \mathscr{C}_{x'}+\mathscr{D}_{x'}$ and define $(\gamma+\delta)_{x'} := \gamma_{x'}+\delta_{x'}$.

Composing with (ϕ,μ'_x) gives $(S+T,\mathscr{C}+\mathscr{D},\gamma+\delta) \xrightarrow{(\phi(g+h),\theta_x)} (X,\mathscr{A},\mu)$ where

$$\theta_x(E\cap(g^{-1}\phi^{-1}(x)+h^{-1}\phi^{-1}(x)))$$
$$= \int_{\phi^{-1}(x)} (\gamma+\delta)_{x'}(E\cap(g^{-1}(x')+h^{-1}(x')))\,\mathrm{d}\mu'_x(x').$$

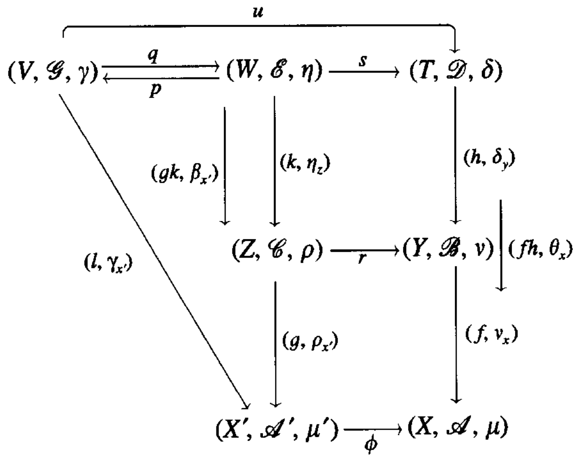
Composing first then forming the coproduct gives $(S,\mathscr{C},\gamma) \xrightarrow{(\phi g,\gamma_x)} (X,\mathscr{A},\mu)$ and $(T,\mathscr{D},\delta) \xrightarrow{(\phi h,\delta_x)} (X,\mathscr{A},\mu)$ which gives $(S+T,\mathscr{C}+\mathscr{D},\gamma+\delta) \xrightarrow{(\phi g+\phi h,\gamma_x+\delta_x)} (X,\mathscr{A},\mu)$. Certainly, $\phi(g+h) = \phi g+\phi h$. We must show the measures are the same:

$$\theta_x((C+D)\cap(g^{-1}\phi^{-1}(x)+h^{-1}\phi^{-1}(x)))$$
$$= \int_{\phi^{-1}(x)} (\gamma+\delta)_{x'}((C+D)\cap(g^{-1}(x')+h^{-1}(x')))\,\mathrm{d}\mu'_x(x')$$
$$= \int_{\phi^{-1}(x)} (\gamma+\delta)_{x'}((C+T)\cap(g^{-1}(x')+h^{-1}(x')))\,\mathrm{d}\mu'_x(x')$$
$$= \int_{\phi^{-1}(x)} \gamma_{x'}(C\cap g^{-1}(x'))\,\mathrm{d}\mu'_x(x') + \int_{\phi^{-1}(x)} \delta_{x'}(D\cap h^{-1}(x'))\,\mathrm{d}\mu'_x(x')$$
$$= \gamma_x(C\cap g^{-1}\phi^{-1}(x)) + \delta_x(D\cap h^{-1}\phi^{-1}(x))$$
$$= (\gamma+\delta)_x(C+D\cap(\phi(g+h))^{-1}(x)).$$ □

5.2. *Beck condition*

Theorem 5.1. *$\sum$ satisfies the Beck condition. More precisely, given $\phi \in$ __M0R__ and f and $h \in$ __Disint__, $\phi^*(\sum_f(h)) \cong \sum_{\phi^*(f)}(r^*(h))$ where $\cong$ is interpreted as measure equivalence.*

Proof. Consider the diagram:



$g = \phi^*(f)$, $k = r^*(h)$, so $gk = \sum_{\phi^*(f)}(r^*(h))$ and $l = \phi^*(\sum_f(h))$. p and q form a measurable isomorphism which respects $\mathscr{G}_{x'}$, $\mathscr{H}_{x'}$, $\gamma_{x'}$, and $\beta_{x'}$ (which implies p and q respect γ and η since these are disintegrations). By *respects*, we mean for each $x' \in X'$, $\beta_{x'}(q^{-1}(G) \cap k^{-1}g^{-1}(x')) = \gamma_{x'}(G \cap l^{-1}(x'))$ and the corresponding equality for p. $\square$

First note that p and q already respect $(V, \mathscr{G})$ and $(W, \mathscr{E})$ (as before, enumerate cases). Explicitly, $W = \{(t,z) \mid h(t) = r(z)\} = \{(t,y,x') \mid h(t) = r(y,x') = y,\ \phi(x') = f(y)\}$, $V = \{(t,x') \mid \phi(x') = fh(t)\}$, $p(t,x') = (t,h(t),x')$, and $q(t,y,x') = (t,x')$. Now, fix $x' \in X'$.

Lemma 5.1. $\beta_{x'}(q^{-1}(G) \cap k^{-1}g^{-1}(x')) = \gamma_{x'}(G \cap l^{-1}(x'))$.

Proof. For brevity, we will only check the case when G is a "measurable rectangle": $G = l^{-1}(A') \cap u^{-1}(D)$. The other calculations are similar.

$$\begin{aligned}
&\beta_{x'}(q^{-1}(l^{-1}(A') \cap u^{-1}(D)) \cap k^{-1}g^{-1}(x')) \\
&= \beta_{x'}(q^{-1}l^{-1}(A') \cap q^{-1}u^{-1}(D) \cap k^{-1}g^{-1}(x')) \\
&= \int_{g^{-1}(x')} \eta_z(k^{-1}g^{-1}(A') \cap s^{-1}(D) \cap k^{-1}g^{-1}(x'))\, \mathrm{d}\rho_{x'}(z) \\
&= \int_{g^{-1}(x')} \delta_{r(z)}(D \cap h^{-1}(r(z))) \cdot \chi_{g^{-1}(A')}\, \mathrm{d}\rho_{x'}(z)
\end{aligned}$$

and

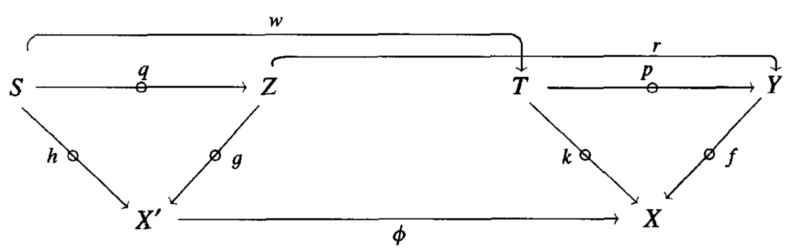
$$\gamma_{x'}(l^{-1}(A') \cap u^{-1}(D) \cap l^{-1}(x')) = \theta_{\phi(x')}(D \cap h^{-1}f^{-1}(\phi(x'))) \cdot \chi_{A'}$$
$$= \int_{f^{-1}(\phi(x'))} \delta_y(D \cap h^{-1}(y))\, \mathrm{d}\nu_{\phi(x')}(y) \cdot \chi_{A'}.$$

Put $a(y) := \delta_y(D \cap h^{-1}(y))$, then $\beta_{x'}(\cdots) = \int_{f^{-1}(\phi(x'))} a(y)\, \mathrm{d}\nu_{\phi(x')}(y) \cdot \chi_{A'}$ and $\gamma_{x'}(\cdots) = \int_{g^{-1}(x')} a(r(z)) \cdot \chi_{g^{-1}(A')}\, \mathrm{d}\rho_{x'}(z)$. We must show that the two integrals are the same. As usual, we build up the proof by looking at characteristic functions, simple functions, and increasing limits of simple functions. The interesting case is the last one. Let $t_n(y) \uparrow a(y)$. Then $t_n(r(z))$ is a sequence of simple functions increasing to $a(r(z))$ (1. $t(y)$ simple $\Rightarrow t(r(z))$ simple: $t(y) = \sum_{i=1}^{m} b_i \chi_{B_i} \Rightarrow t(r(z)) = \sum_{i=1}^{m} b_i \chi_{r^{-1}(B_i)}$ and 2. $t_n(y) \uparrow a(y) \Rightarrow t_n(r(z)) \uparrow a(r(z))$: that the limit works is obvious; for increasing, suppose $a(y) \geq t_n(y)$ *a.a.* y, then $a(r(z)) \geq t_n(r(z))$ *a.a.z*, since $r \in$ **M0R**). With these facts in mind and using the monotone convergence theorem,

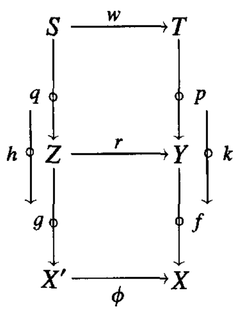
$$\begin{aligned}
\beta_{x'}(\cdots) &= \int_{f^{-1}(\phi(x'))} \lim t_n(y)\, \mathrm{d}\nu_{\phi(x')}(y) \cdot \chi_{A'} \\
&= \lim \int_{f^{-1}(\phi(x'))} t_n(y)\, \mathrm{d}\nu_{\phi(x')}(y) \cdot \chi_{A'} \\
&= \lim \int_{g^{-1}(x')} t_n(r(z)) \cdot \chi_{g^{-1}(A')}\, \mathrm{d}\rho_{x'}(z) \\
&= \int_{g^{-1}(x')} \lim t_n(r(z)) \cdot \chi_{g^{-1}(A')}\, \mathrm{d}\rho_{x'}(z) \\
&= \int_{g^{-1}(x')} a(r(z)) \cdot \chi_{g^{-1}(A')}\, \mathrm{d}\rho_{x'}(z) \\
&= \gamma_{x'}(\cdots). \quad \square
\end{aligned}$$

5.3. Indexed categories

In Section 3.3, it was noted that we may horizontally paste squares. The Beck condition essentially tells us that we may vertically paste squares. As an application: for a fixed ϕ, ϕ^* is a functor. By rearranging the diagram (below, ——o—— denotes disintegration and ———— denotes M0R)



to



we see immediately how to define $q = \phi^*(p)$ to get a functor $\phi^* : \underline{\mathbf{Disint}}/X \to \underline{\mathbf{Disint}}/X'$. Combining this with $1^* \cong 1$ and $(\phi\psi)^* \cong \phi^*\psi^*$, there is a pseudo-functor

$$(\ )^* : \underline{\mathbf{M0R}}^{\mathrm{op}} \to \underline{\mathbf{CAT}}$$

whose object function is $X \mapsto \underline{\mathbf{Disint}}/X$, and so, we get an indexed category $\underset{\sim}{\mathbf{Disint}}$.

Remark. (1) If p is merely measurable, then so is q (these are pullbacks in $\underline{\mathbf{Mble}}$). There is another indexed category (of course, in this case, w and r are merely measurable as well).

(2) In the introduction, it was noted that $\underline{\mathbf{Disint}}$ has a "built-in self-indexing". The above makes this vague phrase more precise.

6. HF/X

6.1. Preamble

We have set up substitution machinery for $\underline{\mathbf{Disint}}$. In this section, we provide an application to operator theory. In [6], we began a program to study the direct integral of Hilbert spaces (see [2] for exposition) in the context of indexed category theory of [4]. Formally, it has a coproduct-like nature and a measure-indexed nature. The idea, then, is to use abstract indexing by measure spaces to put this and similar constructions on a firm, categorical footing. In essence, we want to interpret the picture

$$(\underline{\mathbf{Hilb}})^X \underset{\phi^*}{\overset{\int_\phi^\otimes}{\rightleftarrows}} (\underline{\mathbf{Hilb}})^Y$$

It seems appropriate, from the point of view of analysis, to have $\phi \in \underline{\mathbf{M0R}}$ for ϕ^* (there are many reasons for this but, as an example, *almost everywhere equality* is a common occurrence in measure theory and measure zero reflecting functions are precisely those which are compatible). To construct a useful generalization of the ordinary direct integral $\int_\phi^\oplus$, ϕ must be a disintegration. A good notion of $\underline{\mathbf{Hilb}},^X$ the category of X-families of Hilbert spaces, must be found (incidentally, Hilbert spaces are assumed to be separable). We must reiterate: we do not have classical indexed category theory in the sense of [1] or [4] (our base categories **M0R** and **Disint** do not have products). There are examples to be studied, however, so we wish to approximate the situation as best as possible.

In [6], we put forth the approximation $\underline{\mathbf{Hilb}}^X = \underline{\mathbf{Hilb}}(MEAS(X))$ (i.e., Hilbert space objects in a certain sheaf category constructed from a topos). In this section, we provide another approximation, essentially the *local homeomorphisms* idea. First note that the correspondence of local homeomorphisms with sheaves in topology does not work for measure theory. Indeed, even naively translating topological local homeomorphisms to *measurable local homeomorphisms* (replacing continuous with measurable) leads to problems of triviality (see the introductory remarks of [5]). But, an interesting fragment can be kept.

In the next section, we will introduce a category, HF/X, of (measurable) Hilbert families over an $X \in \underline{\mathbf{Disint}}$. Essentially, we want a measurable (or measure) space over X whose fibres are Hilbert spaces. Before listing the axioms for an HF/X, we will end this section by defining the category $\underline{\mathbf{Mble}}/X$ and describing what should be thought of as the complex numbers in HF/X (to provide a motivational example).

Definition 6.1. Let $(X, \mathscr{A}, \mu) \in \underline{\mathbf{Disint}}$ be fixed. The category $\underline{\mathbf{Mble}}/X$ has as objects

$$\begin{array}{c} (Y, \mathscr{B}) \\ \Big\downarrow f \\ (X, \mathscr{A}, \mu) \end{array}$$

and as morphisms measurable $(Y, \mathscr{B}) \xrightarrow{t} (Y', \mathscr{B}')$ which make the evident triangle commute (i.e. the slice category but over the space X considered as a measurable space).

Notation 1. *We suppress mention of σ-algebras and measures if no confusion can arise. Furthermore, for space considerations, we sometimes write the objects of slice categories sideways.*

A particular object of $\underline{\mathbf{Mble}}/X$ is $(\mathbf{C} \times X, \text{Borel} \times \mathscr{A}) \xrightarrow{p_2} (X, A, \mu)$ where p_2 denotes projection onto the second factor. There is a measurable operation

$$\begin{array}{ccc} X & \xrightarrow{\ulcorner 0 \urcorner} & \mathbf{C} \times X \\ & {\scriptstyle 1} \searrow \quad \swarrow {\scriptstyle p_2} & \\ & X & \end{array}$$

given by $x \mapsto (0,x)$ and other operations (defined over X):

$$\begin{aligned}&\lceil 1\rceil : X \to \mathbf{C}\times X;\; x \mapsto (1,x),\\ &+ : (\mathbf{C}\times X)\times_X(\mathbf{C}\times X) \to \mathbf{C}\times X;\; ((c,x),(c',x)) \mapsto (c+c',x),\\ &\cdot : (\mathbf{C}\times X)\times_X(\mathbf{C}\times X) \to \mathbf{C}\times X;\; ((c,x),(c',x)) \mapsto (cc',x),\\ &- : \mathbf{C}\times X \to \mathbf{C}\times X;\; (c,x) \mapsto (-c,x)\end{aligned}$$

and

$$(^{-}) : \mathbf{C}\times X \to \mathbf{C}\times X;\; (c,x) \mapsto (\bar{c},x).$$

With these operations, $\mathbf{C}\times X \xrightarrow{p_2} X$ is a commutative $*$-algebra (scalar multiplication is the same as multiplication). It satisfies the axiom of non-triviality (see [3]). In fact, it is a geometric field (a statement which still makes sense in $\underline{\mathbf{Mble}}/X$ even though it is not a topos). Here, the group of units is $U = \mathbf{C}\backslash\{0\}\times X \to X$ and $\lceil 0\rceil = \{0\}\times X$ and $U + 0 = (\mathbf{C}\backslash\{0\}\times X) + (\{0\}\times X) \simeq \mathbf{C}\times X$ (over X) via $((c,x),1) \mapsto (c,x)$ and $((0,x),2) \mapsto (0,x)$. Thus, $\mathbf{C}\times X$ is a geometric field in $\underline{\mathbf{Mble}}/X$.

6.2. *HF*/*X*

An object of HF/X is $(Y,\mathscr{B}) \xrightarrow{f} (X,\mathscr{A},\mu) \in \underline{\mathbf{Mble}}/X$ subject to three axioms:

Axiom (a). $Y_x = f^{-1}(x)$ is a separable Hilbert space for each $x \in X$.

Part of the data for axiom (a) provides us with maps relevant for algebra and topology like those for $\mathbf{C}\times X$. In more precise terms, we have maps, defined over X: $X \xrightarrow{\lceil 0\rceil} Y, \lceil 0\rceil(x) = 0_x \in Y_x$; $Y \xrightarrow{-(\ )} Y, -(y_x) = -_x\, y_x$; $Y\times_X Y \xrightarrow{+} Y, +(y,y',x) = y +_x y'$; $(\mathbf{C}\times X)\times_X Y \xrightarrow{\cdot} Y, \cdot((c,x),y_x) = c\cdot_x y_x$; and $Y\times_X Y \xrightarrow{\langle -|-\rangle} \mathbf{C}\times X, \langle -\mid -\rangle(y,y',x) = (\langle y\mid y'\rangle_x, x)$. These make Y into a $\mathbf{C}\times X$-vector space with an $\mathbf{R}^{\geq 0}\times X$-valued norm satisfying the parallelogram law.

Axiom (b). The maps in the above paragraph are all measurable. That is, $(Y,\mathscr{B})$ is a $(\mathbf{C}\times X$, Borel $\times\,\mathscr{A})$-inner product space in $\underline{\mathbf{Mble}}/X$.

Definition 6.2. A *sequence* in Y is a measurable map over $X, \mathbf{N}\times X = X^*(N) \xrightarrow{s} Y$.

Remark. $\mathbf{N}\times X \xrightarrow{s} Y$ over X is an ordinary sequence of measurable maps $X \xrightarrow{s_n} Y$ over X, a positive real, $\varepsilon \in \mathbf{R}^{>0}\times X$, is a measurable $X \xrightarrow{\varepsilon} \mathbf{R}^{>0}$, and a natural number is a measurable $X \xrightarrow{N} \mathbf{N}$.

Definition 6.3. A sequence, s_n, is said to *converge* if there is an $s \in Y$ (which means a measurable section $s : X \to Y$) such that $\forall \varepsilon(x) \in \mathbf{R}^{>0}\times X$, $\exists N(x) \in \mathbf{N}\times X$ such that $\forall n(x) \geq N(x), \|s_{n(x)} - s\|(x) < \varepsilon(x)$.

Remark. (1) $<$ and $\leq$ are interpreted as being everywhere as opposed to almost everywhere.

(2) A *Cauchy sequence* is defined in a similar manner. Likewise, *completeness* of Y has an obvious definition.

Completeness of Y is not enough to make substitution work. We will need stability under substitution squares:

Axiom (c). Y is stably complete.

This means, for all $X' \xrightarrow{\phi} X \xleftarrow{f} Y$, and for all ϕ-sequences (i.e. measurable s's such that

$$\begin{array}{ccc} \mathbf{N} \times X' & \xrightarrow{s} & Y \\ {\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle f} \\ X' & \xrightarrow[\phi]{} & X \end{array}$$

commutes) ϕ-Cauchy (i.e. $\forall \varepsilon(x') \in \mathbf{R}^{>0} \times X'$, $\exists N(x') \in \mathbf{N} \times X'$ such that $\forall n(x')$, $m(x') \geq N(x'), \|s_{n(x')} - s_{m(x')}\|(\phi(x')) < \varepsilon(x')$) implies ϕ-convergent (with a similar definition).

Remark. (1) $\|\cdot\|$ is a measurable function $Y \to \mathbf{R} \times X$ over X and for each section $X \xrightarrow{s} Y$, $\|s\|$ is a measurable function $X \to \mathbf{R}$.

(2) As we shall see below stable completeness implies that each substitution object $(Z, \mathscr{C})$, is complete. In particular, the completeness of $(Y, \mathscr{B})$ is a special case with $\phi = 1$.

Definition 6.4. A morphism of *HF/X* is a measurable T

$$\begin{array}{ccc} (Y, \mathscr{B}) & \xrightarrow{T} & (Y', \mathscr{B}') \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & X & \end{array}$$

making the triangle commute such that each $T_x : Y_x \to Y'_x$ is a bounded linear map and $\|T_x\|_{Y_x}$ is bounded over $x \in X$.

Remark. There are actually three categories relevant to this work (the first two have obvious objects and morphisms): *PreHilb/X*, *Complete/X*, and *HF/X* = *StablyComplete/X*.

We end this section by discussing change of base in relation to Hilbert families. A result which we will find useful is:

Lemma 6.1. *Let H be a complete metric space with dense sequence $\{h_i\}$. Then the σ-algebra of Borel sets is generated by the open balls of rational radius about the h_i's.*

Proof. Every open set is a countable union of such open balls. □

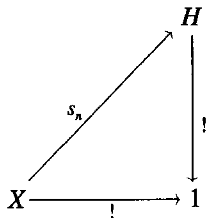
Let us consider the special case $HF/1$ first. Specifically, we will describe an adjunction

$$\underline{\mathbf{Hilb}} \overset{F}{\underset{I}{\leftrightarrows}} HF/1.$$

Define $I(H)=(H,\text{Borel}) \xrightarrow{!} (1,2,\text{counting})$. Axioms (a) and (b) are satisfied (the relevant maps are all continuous so are all Borel measurable).

Proposition 6.1. *$I(H)$ satisfies axiom (c).*

Proof. Let



be a !-Cauchy !-sequence. We claim that $s_n(x)$ is pointwise Cauchy for each x. Fix x_0 and let $\varepsilon > 0$ be given. Put $\varepsilon(x)=\lceil\varepsilon\rceil$ then there is an $N(x)$ such that $\forall n(x), m(m) \geq N(x), \|s_{n(x)}-s_{m(x)}\| < \varepsilon$. Now, let $N=N(x_0)$ and $p,q \geq N$. If we set $p(x)=\max\{\lceil p\rceil, N(x)\}$ and $q(x)=\max\{\lceil q\rceil, N(X)\}$, then $p(x)$ and $q(x)$ are measurable, $p(x),q(x) \geq N(x)$, $p(x_0)=p$, and $q(x_0)=q$, so $\|s_p - s_q\| < \varepsilon$. And so, $s_n(x_0)$ is Cauchy.

Since H is complete, there is an $s(x)$ such that $s_n(x) \to s(x)$ for each x. In addition, $\|s_n(x)\| \to \|s(x)\|$ since $\mathbf{R}$ is complete and $\|\cdot\|$ is continuous. The pointwise limit in $\mathbf{R}$ of measurable functions yields a measurable function. That is, $\|s(x)\|$ is measurable. But, as a consequence of Lemma 6.1, $s(x)$ is measurable as well (each $s^{-1}(B(0,r))=s^{-1}\{h\in H \mid \|h\|<r\}=\{x\in X \mid \|s(x)\|<r\}\in\mathscr{A}$ since $\|s(x)\|$ is measurable; then use the measurable translation (= adding a fixed vector) to get other open balls).

To exhibit !-completeness of $H \xrightarrow{!} 1$, we need only show $s_n(x) \to_{p.w.} s(x) \Rightarrow s_n \to s$ in the sense of HF/X. Let $\varepsilon(x)$ be given. Suppose first that it is constantly ε. For each x, there is an N such that $\|s_n(x)-s(x)\| < \varepsilon$ for all $n \geq N$. Put $N(x)=\min\{N \mid \|s_n(x)-s_m(x)\| < \varepsilon \forall n \geq N\}$. All we need to show is that $N(x)$ is measurable. But

$N^{-1}(k)=A_k\backslash A_{k-1}$ where $A_k=\bigcup_{t=k}^{\infty}\{x|\|s_n(x)_s(x)\|<\varepsilon\}$ is measurable. A general $\varepsilon(x)$ can be approximated below by simple functions. Apply the above case repeatedly to arrive at the inequality for a simple function and hence the inequality for a general ε. □

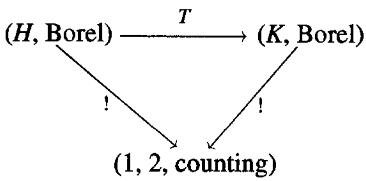
Remark. In essence, $I(H)\in HF/1$ is *complete* iff it is *stably complete.* We actually have shown one direction for Cauchy and the other direction for convergence but the rest is similar. It is important to note that this does not generalize to HF/X, however. That is, fibrewise completeness $\not\Leftrightarrow$ stable completeness; neither direction holds. (For $\Rightarrow$, we cannot assume s is measurable in general (Lemma 6.1 is special); for $\Leftarrow$, we cannot take a sequence Cauchy in one, fixed fibre and produce a global Cauchy sequence since the fibres "are of global measure zero", for example,

$$s_n(x)=\begin{cases} s_n(x_0), & x=x_0,\\ 0 & \text{else}\end{cases}$$

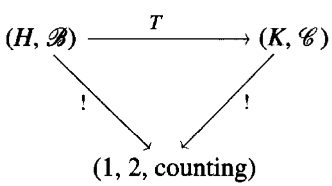
is essentially the 0 function; of course, if x_0 is an atom, this works.) For this reason, we impose both completeness conditions. Both together are strictly stronger than either one separately.

Proposition 6.2. *I is full.*

Proof. A morphism, $H\xrightarrow{T}K\in\underline{\mathbf{Hilb}}$, yields a morphism



(T is continuous so it is Borel measurable). Furthermore, a

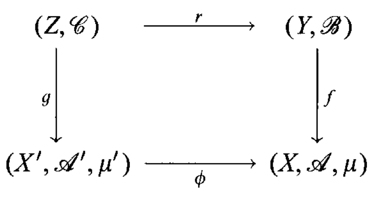


in $HF/1$ is, in particular, a bounded linear transformation from H to K. □

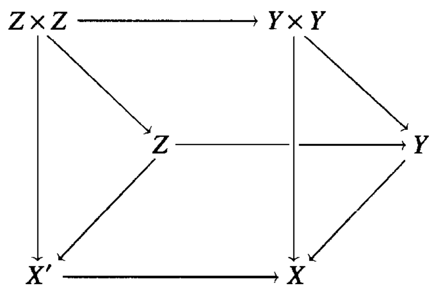
Proposition 6.3. *I has a left adjoint F.*

Proof. Axiom (b) for $(H,\mathscr{B})\xrightarrow{!}(1,2,\text{counting})$ says, in particular, $\|\cdot\|$ and translation are measurable with respect to $\mathscr{B}$, and so, $\mathscr{B}$ must contain the Borels. Thus, forgetting the measurable structure on $(H,\mathscr{B})$ provides a left adjoint F to I. □

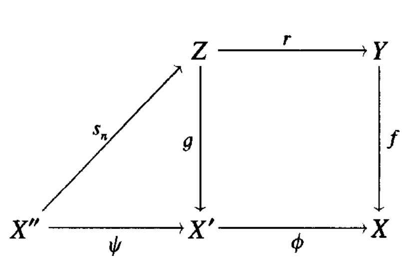
Next, we discuss the general situation. Suppose $(X', \mathscr{A}', \mu') \xrightarrow{\phi} (X, \mathscr{A}, \mu)$ is in **M0R**. In



$Z_{x'} = g^{-1}(x') = Y_{\phi(x')}$ is a Hilbert space. The operations of arithmetic and the inner product are measurable when "pulled back" along ϕ. For example, $X' \xrightarrow{\lceil 0 \rceil} Z$ is $x' \mapsto 0_{x'} = 0_{\phi(x')}$ which is just the composition of 0_Y an ϕ. For addition, the relevant picture is



$Z = \sum_{x' \in X'} Y_{\phi(x')}$ and the measurable $+_Y$ yields a measurable $+_Z$ given by $(y, x') + (y', x') = (y +_{\phi(x')} y', x')$. For stable completeness, we must show $(Z, \mathscr{C}) \xrightarrow{g'} (X' \mathscr{A}', \mu')$ is ψ-complete for all $X'' \xrightarrow{\psi} X'$. Let



be a ψ-sequence in Z. Compose with r to get the $\phi\psi$-sequence $t_n = rs_n$ in Y. Let $\varepsilon(x'') \in \mathbf{R}^{>0} \times X''$ be given. Then $\|rs_{n(x'')} - rs_{m(x'')}\|_Y(\phi\psi(x'')) < \varepsilon(x'')$ iff $\|s_{n(x'')} - s_{m(x'')}\|_Z(\psi(x'')) < \varepsilon(x'')$ since $Z_{x'} = Y_{\phi(x')}$, so, in particular, $Z_{\psi(x'')} = Y_{\phi\psi(x'')}$, and the two norms mean the same thing. Thus, s_n is ψ-Cauchy iff t_n is $\phi\psi$-Cauchy and similarly for convergence. Since Y is $\phi\psi$-complete for all ψ, Z is ψ-complete for all ψ.

Pulling back a $Y \xrightarrow{T} Y'$ in HF/X yields a $Z \xrightarrow{T^*} Z'$ in HF/X'. Pseudo-functorial substitution restricts to Hilbert families. This discussion provides an important example:

Example. For each $H \in \underline{\mathbf{Hilb}}$, $\Delta I(H) = (H \times X, \text{Borel} \times \mathscr{A}) \xrightarrow{p_2} (X, \mathscr{A}, \mu)$ is an object of HF/X. These are to be thought of as the constant X-families. Δ is a functor $\underline{\mathbf{Hilb}} \to HF/X$.

6.3. *Direct integral and HF/X*

We next construct the direct integral $HF/X \xrightarrow{\int^{\oplus}} \underline{\mathbf{Hilb}}$. For $Y \in HF/X$, define

$$\int^{\oplus} (Y, \mathscr{B}) \xrightarrow{f} (X, \mathscr{A}, \mu) := \left\{ s : X \to Y \mid s \text{ measurable}, \ fs = 1_x, \right.$$

$$\left. \text{and} \int \|s(x)\|^2 \, \mathrm{d}\mu < \infty \right\} \Big/ \sim,$$

with $s \sim s'$ iff $\mu\{x \mid s(x) \neq s'(x)\} = 0$. This is sometimes written as $\int^{\oplus} Y$. Furthermore, define:

$$\lceil 0 \rceil(x) = 0_x, \qquad (-s)(x) = -_x s(x), \qquad (s + s')(x) = s(x) +_x s'(x) \quad \text{and}$$
$$(\alpha \cdot s)(x) = \alpha \cdot_x s(x).$$

With these definitions, $\int^{\oplus} Y$ is a $\mathbf{C}$-vector space.

Remark. If $\alpha(x) \in L^{\infty}(X, \mathbf{C})$, then modifying scalar multiplication to $(\alpha \cdot s)(x) = \alpha(x) \cdot_x s(x)$ makes $\int^{\oplus} Y$ into an $L^{\infty}(X, \mathbf{C})$-module.

Define an inner product on $\int^{\oplus} Y$ as

$$\langle s | s' \rangle = \int \langle s(x) | s'(x) \rangle_x \, \mathrm{d}\mu$$

which gives a norm $\|s\|_2 = \int \|s(x)\|_x^2 \, \mathrm{d}\mu$. Since functions which are equal almost everywhere are considered equal, $\|s\| = 0 \Rightarrow s = 0$.

Theorem 6.1. *$\int^{\oplus} Y$ is complete.*

Proof. We mimic the classical proof (see [2]). For $\|\cdot\|_2$ – Cauchy sequence s_n, choose a subsequence (also called s_n) such that $\sum_{n=1}^{\infty} \|s_{n+1} - s_n\| < \infty$. In particular, $\sum_{n=1}^{\infty} \|s_{n+1}(x) - s_n(x)\|_x < \infty$ for all $x \notin N$ where N is some measurable set of measure zero.

For $x \notin N$, $s_1(x) + \sum_{n=1}^{\infty} (S_{n+1}(x) - s_n(x))$ converges to an $s(x) \in Y_x$ (Y_x is a Hilbert space) and $f \circ s(x) = x$ since $s(x) \in Y_x$. For $x \in N$, put $s(x) = 0_x$. We must show that $s(x)$ is measurable and square integrable. But, $s(x)$ is the limit, almost everywhere, of

$s_{(p)}(x) := \sum_{n=1}^{\infty}(s_{n+1}(x) - s_n(x))$ each of which is measurable. Furthermore,

$$\int \|s(x)\|_x^2 \,\mathrm{d}\mu(x) \le \int \|s_1(x)\|_x^2 \,\mathrm{d}\mu(x) + \sum_{n=1}^{\infty} \int \|s_{n+1}(x) - s_n(x)\|_x^2 \,\mathrm{d}\mu(x)$$

$$= \int \|s_1(x)\|_x^2 \,\mathrm{d}\mu(x) + \sum_{n=1}^{\infty} \|s_{n+1} - s_n\|_2 < \infty. \qquad \square$$

Remark. This is actually get an object of $HF/1 : (\int^{\oplus} Y, \text{Borel})$.

For $(Y \xrightarrow{f} X) \xrightarrow{T} (Y' \xrightarrow{f'} X)$ in HF/X, define

$$\int^{\oplus} T \colon \int^{\oplus} Y \to \int^{\oplus} Y'; \; s \mapsto Ts; \quad Ts(x) = T_x s(x).$$

Now, $T(s + s')(x) = T_x s(x) +_x Ts'(x) = Ts(x) + Ts'(x)$ and $T(\alpha s)(x) = T_x \alpha \cdot_x s(x) = \alpha \cdot_x T_x s(x) = \alpha \cdot T(s)(x)$. Since $\|T_x\|_x$ is bounded (across x), then

$$\int \|Ts(x)\|_x^2 \,\mathrm{d}\mu = \int \|T_x s(x)\|_x^2 \,\mathrm{d}\mu \le \int \|T_x\|_x^2 \|s(x)\|_x^2 \,\mathrm{d}\mu \le k \int \|s(x)\|_x^2 \,\mathrm{d}\mu < \infty.$$

And so, there is a functor: $\int^{\oplus} : HF/X \to \underline{\mathbf{Hilb}}$.

Remark. $\int^{\oplus} \Delta H = \int^{\oplus} H \times X \xrightarrow{p_2} X = \{s : X \to H \times X \mid s$ measurable $p_2 s = 1$, and $\int \|s(x)\|_x^2 \,\mathrm{d}\mu < \infty\} = L^2(X;H)$ (here we abuse notation and call $\Delta H = \Delta IH$).

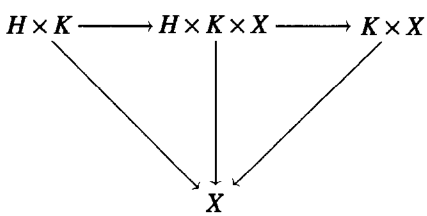
Let us expand on this remark. $L^2(X;H)$ is functorial in H. Given a bounded linear map $F : H \to H'$, we get a map, $L^2(X;H) \xrightarrow{L^2(X;F)} L^2(X;H')$, $f \mapsto \hat{f}(x) = Ff(x)$. Since F is continuous, $Ff(x)$ is measurable and $\int \|Ff(x)\|^2 \,\mathrm{d}\mu \le \int \|T\|^2 \|f(x)\|^2 \,\mathrm{d}\mu < \infty$.

There is a map $H \xrightarrow{T} L^2(X;H) h \mapsto \lceil h \rceil$ (recall, $\mu(X) < \infty$) which is linear and bounded ($\|Th\| = (\int \|h\|^2 \,\mathrm{d}\mu)^2 = \|h\| \mu(X)^{1/2}$ so $\|T\| = \mu(X)^{1/2}$) and natural in H. The natural transformation T is, in general, not an isomorphism (unless $X = 1$).

Two interesting properties of Δ are:

Proposition 6.4. (a) $\Delta(H \oplus K) = \Delta(H \times K) = \Delta(H) \times \Delta(K)$ *and* (b) $\Delta(1) = 1$.

Proof. (a) One must simply show that



with the evident projections is a product diagram.

(b) $\Delta(1) = 1 \times X \xrightarrow{p_2} X \simeq X \xrightarrow{1} X$. $\square$

In view of the fact that $0 = 1$ in **Hilb** (both are the one-point Hilbert space) and $0 \neq 1$ in HF/X (0 is $\emptyset \hookrightarrow X$ and 1 is $X \to X$), we have:

Corollary. Δ does not preserve 0 and Δ does not have a right adjoint.

Note that $\int^{\oplus}$ is not left adjoint to Δ. The unit would be

$$\begin{array}{ccc} H & \longrightarrow & (\int^{\oplus} H) \times X \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle p_2} & \\ & X & \end{array}$$

$h \in H_x$ gets sent to the function in $\int^{\oplus} H$ that sends $x \mapsto h$ and everything else to 0. In the case X is a finite set with counting measure, everything works. But, if points have measure zero in X, then the function so described is the 0 map (after modding out by a.e. equality) and so there is no *injection*.

Also, the counit would be a map $L^2(X; H) \to H$ and given an L^2-function, there seems to be no *canonical* way of getting an element of H (we would need some sort of "indefinite" integral $h = \int f(x)\, \mathrm{d}\mu$ and a square integrable function is not necessarily integrable).

7. Epilogue

The question of how to generalize the above to get a ϕ-direct integral seems to be quite difficult. We finish with a few remarks on this. Suppose $(X', \mathscr{A}', \mu') \xrightarrow{(\phi, \mu'_x)} (X, \mathscr{A}, \mu)$ is a disintegration. For $(T, \mathscr{D}) \xrightarrow{h} (X', \mathscr{A}', \mu')$, put

$$\left(\int_{\phi}^{\oplus} (T, \mathscr{D})\right)_x := \left\{ s : \phi^{-1}(x) \to T \mid s \quad \text{a measurable } \phi\text{-section}, \right.$$

$$\left. \int_{\phi^{-1}(y)} \|s(x')\|^2 \, \mathrm{d}\mu'_x(x') < \infty \right\} \Big/ \sim$$

where $s \sim s'$ iff $\mu'_x\{x' \in \phi^{-1}(x) \mid s(x') \neq s'(x')\} = 0$. Equivalently, we could take global measurable sections, $s : X \to T$, with the same $\sim$. Next, take the coproduct to get

$$\sum_{x \in X} \left(\int_{\phi}^{\oplus} (T, \mathscr{D})\right)_x =: Y \xrightarrow{p} X$$

with p the evident projection. There is no obvious way to put a σ-algebra structure, $\mathscr{B}$, on Y (exceptions: $\int_1^{\oplus}(T, \mathscr{D}) = (T, \mathscr{D})$ and $\int_!^{\oplus}(T, \mathscr{D}) = (\int^{\oplus} T, \text{Borel})$). Indeed, this is an interesting open problem.

One idea is to take simply the Borels in each fibre (note: each $(\int_{\phi}^{\oplus}(T, \mathscr{D}))_x$ is a Hilbert space). This would be the σ-algebra of the infinite coproduct (= disjoint union). The problem is that this provides no compatibility across the fibres. Consider the example suggested by the picture:

Each fibre is a Borel set. However, these may slide back and forth in a non-measurable way to produce a globally non-measurable set. The converse is problematic as well: slicing a Borel set does not necessarily produce a Borel set.

In some sense, these are function spaces (a special case is $L^2(X)$ which would work except for the caveat about slicing a Borel just mentioned). A related question, and another idea, then, is how to put a useful σ-algebra structure on a function space. Obvious things such as the *infinite product* structure or the *measurable-measurable* σ-algebra (in analogy to the compact-open topology) do not seem to work (we need, for example, a more appropriate translation of *compact set*).

Our feeling is that disintegrations provide the answer. Some sense needs to be made of statements like "$\mathscr{A} = \int \mathscr{A}_y \, \mathrm{d}\nu(y)$", in the same manner as "$\mu = \int \mu_y \, \mathrm{d}\nu(y)$" and in a way that does not conflict with square integrability. Given a measure, we may disintegrate along slices. But conversely, given slice spaces and gluing them together requires some sort of global compatibility condition. It is possible that this is related to the unsolved "existence of (ordinary) disintegration" problem (as a generalization of the Radon–Nikodym theorem, one may be interested in the question of when a measure space may be disintegrated with respect to another measure space).

Finally, we make two observations. All this works in $\underline{\mathbf{Set}}/X$. For this reason, we believe this is the *correct* notion of direct integral in *HF*/*X*. The difficult part is putting a measurable structure on it. Furthermore, we have not yet been able to employ the full power of the substitution machinery of disintegrations. That is, we should also be able to put a measure structure on all these entities. This would be part of another program: understand the difference between measure theory and topology with respect to slicing and indexing.

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